Report on the GFD 2009 at Woods Hole

principle lectures: Nonlinear waves
project: Laboratory Experiments on Two Coalescing Axisymmetric Turbulent Plumes in a Rotating Fluid

Hiroki Yamamoto (Kyoto Univ.)

GFD Summer Program

- 10 weeks fellowship program at Woods Hole Oceanographic Institution since 1959.
- 2 weeks principle lecture + 8 weeks research project
- Fellows ~ 10 people
- This year's theme "Nonlinear Waves".

Wood Holeってどこ?

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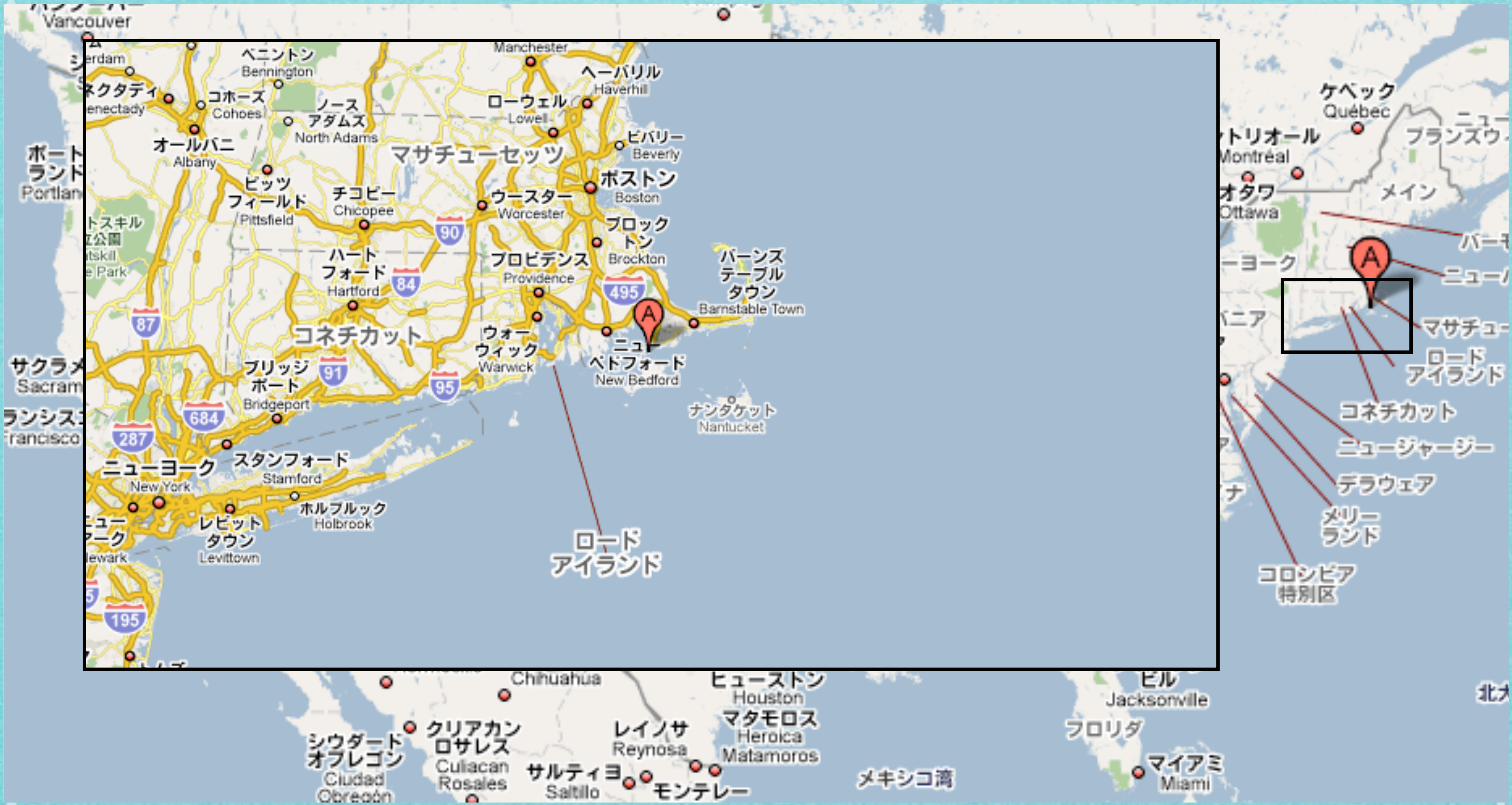
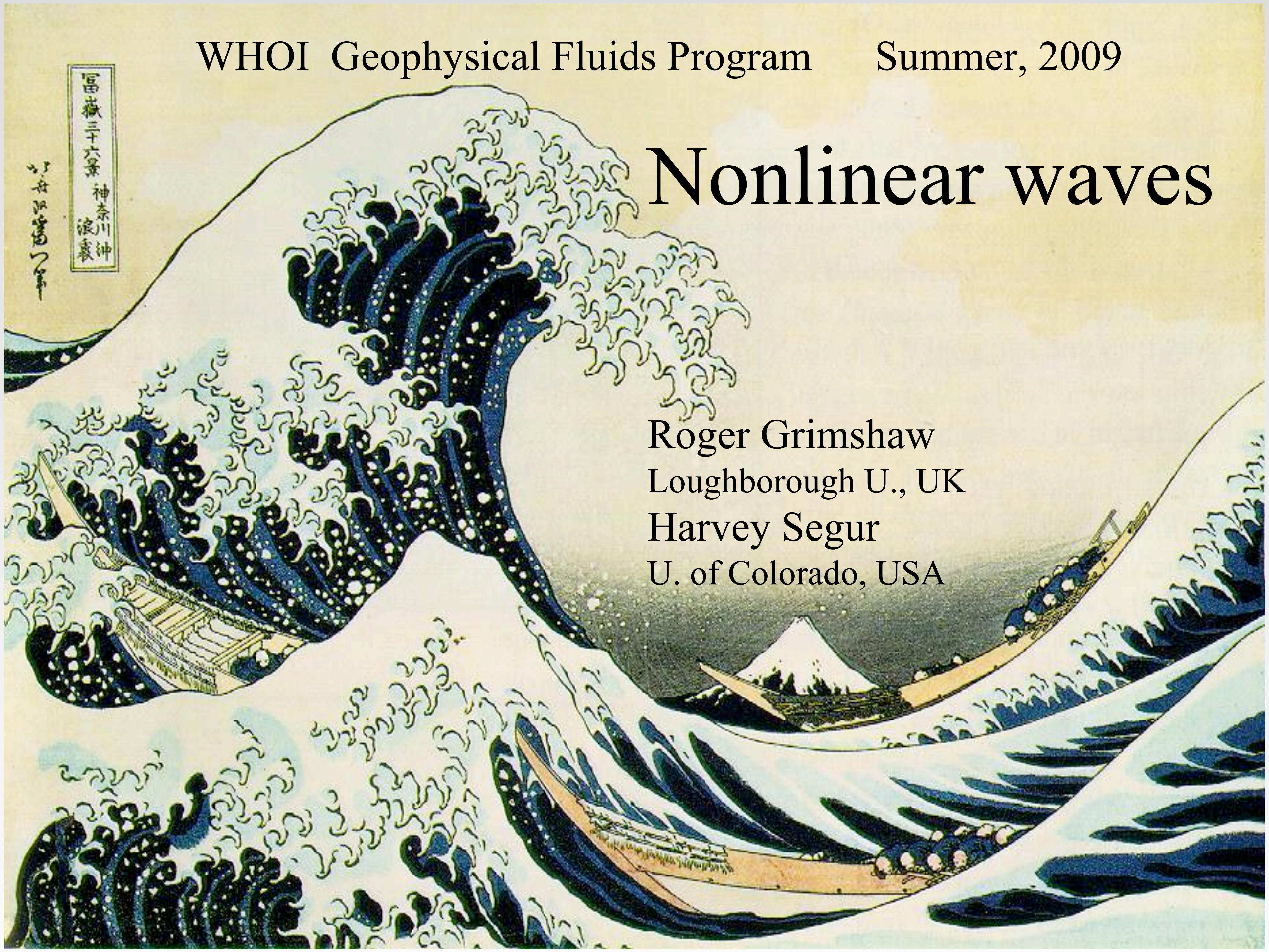


今年の Fellows

名前	国籍	大学	野球のルール
Erinna	USA	カリフォルニア	○
Adrienne	USA	カリフォルニア	○
Andong	中国	ペンステート	×
Yiping	中国	カリフォルニア	×
Alireza	イラン	トロント	×
Michael	オーストラリア	サウスウェールズ	×
Nicolas	フランス	グルノーブル	×
Helene	フランス	エコールノルマル	×
Andrews	イギリス	オックスフォード	×
Daisuke	日本	ケンブリッジ	○
Hiroki	日本	京都	○

Principle lectures



What I expected

- **Wave-mean flow interactions**

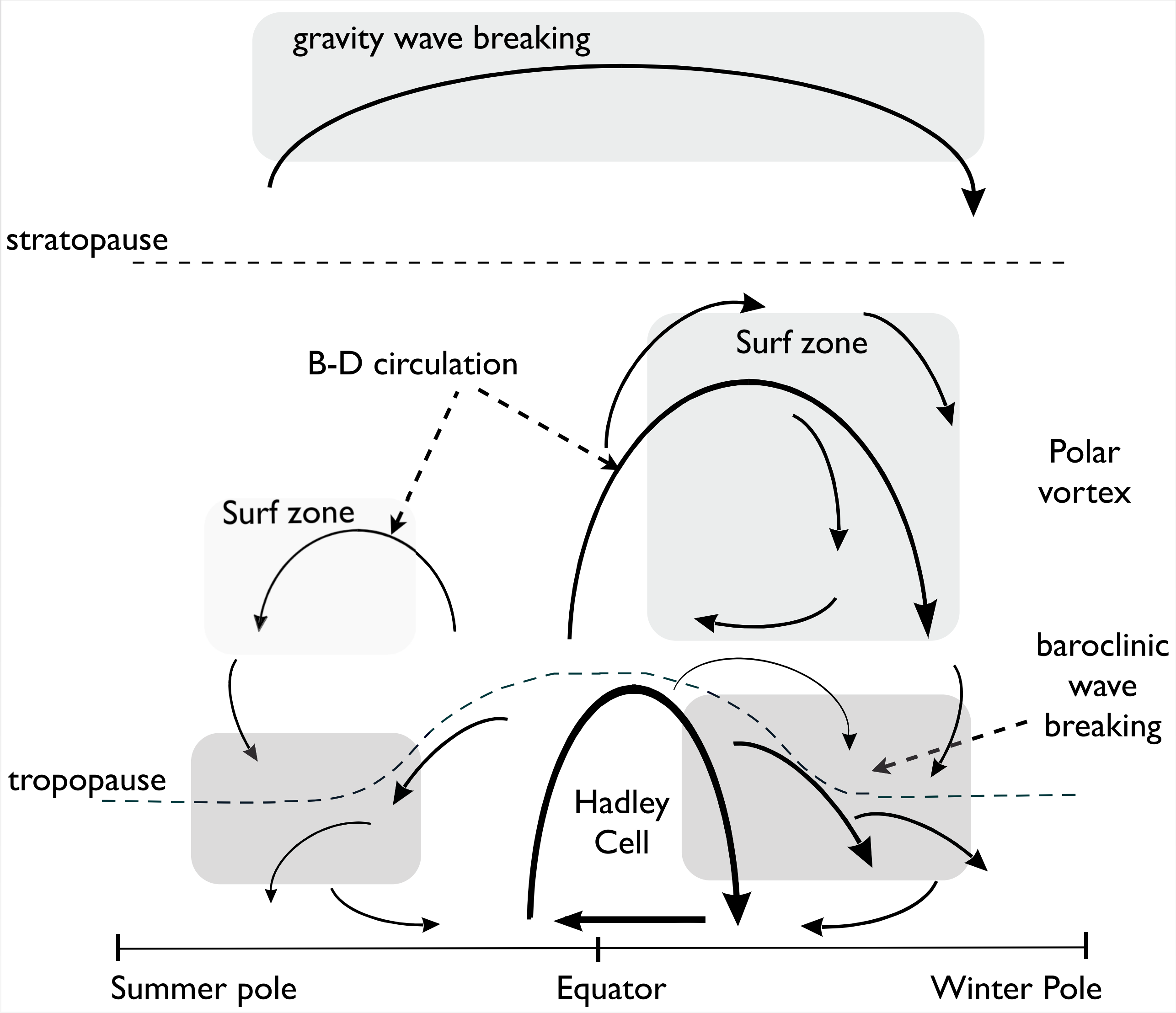


Fig. 13.14 A schema of the residual mean meridional circulation of the atmosphere. The solid arrows indicate the residual circulation (B-D for Brewer Dobson) and the shaded areas the main regions of wave breaking (i.e., enstrophy dissipation) associated with the circulation. In the surf zone the breaking is mainly that of planetary Rossby waves, and in the troposphere and lower stratosphere the breaking is that of baroclinic eddies. The surf zone and residual flow are much weaker in the summer hemisphere. Only in the Hadley Cell is the residual circulation comprised mainly of the Eulerian mean; elsewhere the eddy component dominates.[22]

Vallis (2006)

Actually...

- **Solitary waves** = waves which are isolated and *steadily* propagating pulse.
  = waves which do not break!

Solitary waves in the ocean

Both photos taken in Hawaii by Robert Odom
App. Phys. Lab., U of Washington
see www.amath.washington.edu/~bernard/kp/waterwaves.html

Lecture titles

Harvey Segur's lectures

1. Introduction to water waves

3. Hamiltonian formation: water waves

4. Waves in shallow water

7. Oceanographic Applications

8. The shallow water equations

12. Triad (or 3-wave) resonances

13. Waves on deep water, I

14. Waves on deep water, II

19. The Explosive Instability due to 3-wave of 4-wave mixing

20. Potpourri

Roger Grimshaw's lectures

2. Introduction

5. Derivation of the Korteweg-de Vries Equation

6. Nonlinear waves in a variable medium

9. Internal solitary waves in the ocean

10. Whitham Modulation Theory

11. Transcritical flow over an obstacle

15. Solitary Waves

16. General Solitary Waves

17. Wave-Mean Flow Interaction, Part I

18. Wave-Mean Flow Interaction, Part II

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Therefore, I’ll just introduce the first “Introduction” of Harvey’s lecture and Roger’s lecture.

Harvey’s fist lecture

Introduction to water waves

Why study water waves?

a) Woods Hole Oceanographic Institute

b) For the Program on Nonlinear Waves:

Water waves provide a concrete physical example of a dynamical system rich enough to exhibit many of the mathematical concepts that have been developed in recent years:

linear stability, nonlinear stability

solitons, complete integrability

chaos, sensitive dependence on initial data

singularities, blow-up in finite time

deterministic vs. probabilistic models

c) Water waves evolve on a "human" time-scale, so we can observe many of these concepts in physical experiments

(GFD lecture 1)

Introduction to water waves

Q: What are "water waves"?

A (for my lectures): Waves in the water that you see or feel at the beach or in a boat

(sometimes called "surface water waves")

(GFD lecture 1)

Properties of water waves

- Surface water waves have their maximum displacement at the free surface
- Waves propagate, with little dissipation
  - ask a baby in a bath-tub
  - Snodgrass *et al* (1966)
- Approximately periodic: 0.1 s < T < 25 s.
- Approximate maximum speed: $c = \sqrt{gh}$
- Water waves are “dispersive”:

  Long waves travel faster than short waves
  (for gravity-induced waves)

Ocean waves I am ignoring

- Sound waves (pressure waves) in water

  Speed of sound in water (at 10° C): 1450 m/sec

  Speed of 2004 tsunami: < 200 m/sec

  Pressure waves create initial conditions for surface waves

- Internal waves

  Due to variations in fluid density

  Period of surface waves: seconds

  Period of internal waves: hours

- Inertial waves (including Rossby waves)

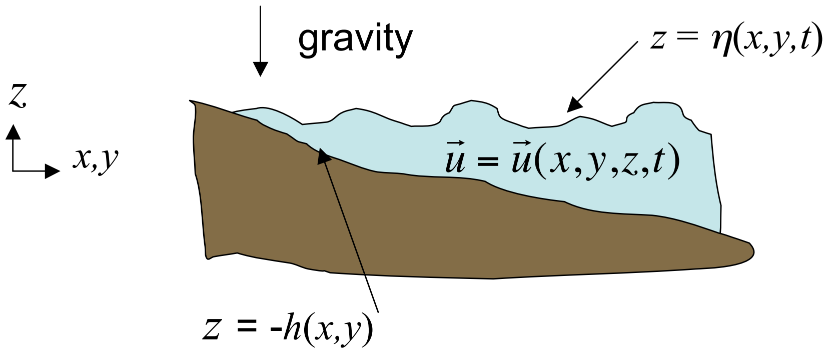
  Due to rotation of earth

  Period of inertial waves ≥ 12 hours

1. Introduction to water waves

Derive the governing equations

(following Stokes, 1847)



Equations of motion

for an irrotational flow, with no forcing from wind:

$$\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi, \qquad \text{on } z = \eta(x,y,t)$$

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{\sigma}{\rho}\nabla\cdot\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\}, \qquad \text{on } z = \eta(x,y,t),$$

$$\nabla^2\phi = 0 \qquad -h(x,y) < z < \eta(x,y,t),$$

$$\partial_z\phi + \nabla\phi\cdot\nabla h = 0, \qquad \text{on } z = -h(x,y).$$

(GFD lecture 1)

Roger’s first lecture

Introduction

Examples of wave motion:

Water waves, atmospheric and oceanic internal waves (gravity waves), sound waves (music), electromagnetic waves (light, radio), elastic waves (earthquakes), etc.

An atmospheric gravity wave train in Northern Australia: the "Morning Glory".

1.1: Linear Waves

For simplicity, assume only one space dimension, and that a typical field variable is $u(x,t)$. Linear waves can then be represented by the Fourier component,

$$u = Real[A \exp(ikx - i\omega t)] \,, \tag{1}$$

where k is the *wavenumber*, ω is the wave *frequency* and A is the wave amplitude, which may also be a function of k. The full solution is obtained by a superposition of such components.

The wave dynamics are determined by the dispersion relation

$$\omega = \omega(k) \,, \tag{2}$$

whose precise form is determined by the physical system under consideration. For instance, for water waves,

$$\omega^2 = gk \tanh(kh) \,, \tag{3}$$

where h is the still water depth, and g is gravity. Here there are two branches of the dispersion relation.

1.2: Linear Waves

An important feature of linear waves is that the dispersion relation captures the full system in Fourier space. That is, if the physical system takes the schematic form

$$D(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x})\, u = 0\,, \quad \text{then} \quad D(\omega, k) = 0\,, \tag{4}$$

whose solutions are the branches $\omega = \omega(k)$. For stable waves, ω is real-valued for all real-valued k. There are two important velocities,

$$\text{Phase Velocity}: \quad c = \frac{\omega}{k}, \quad \text{and} \quad \text{Group Velocity}: \quad c_g = \frac{d\omega}{dk}\,. \tag{5}$$

For a dispersive wave system, they are different. The phase of the wave (e.g. a wave crest) propagates with velocity c, but the wave energy propagates with the velocity c_g. The wave energy E for each Fourier component is typically given by an expression of the form $E = F(k)|A|^2$. For instance, for water waves $E = g|A|^2/2$ where A is the surface elevation above the still-water depth.

1.3: Nonlinear Waves

In general, as a linear dispersive wave system evolves, each Fourier component with wavenumber k propagates with its own group velocity, and so the system disperses. Then nonlinearity, that is the necessity to take account that the amplitude is finite and not infinitesimally small, typically arises in three scenarios.

(1) **Long waves**: Here $k \to 0$. Because the dispersion relation can be made to satisfy the antisymmetry condition $\omega(k) = -\omega(-k)$ (ensuring real-valued solutions), it follows that when also $\omega(0) = 0$, we have that $\omega = c_0 k + O(k^3)$, and so $c_g = c_0 + O(k^2)$, with **weak dispersion**.
(2) **Wave packets**: Here it is assumed that the wave energy is concentrated around a finite wavenumber k_0 say. Consequently, there is again only **weak dispersion**, and approximately the wave group propagates with a constant group velocity $c_{g0} = c_g(k = k_0)$.
(3) **Resonant wave interactions**: Due to nonlinearity, two linear waves with wavenumbers $k_{1,2}$ say, will interact to form another wave with wavenumber $k_0 = k_1 + k_2$. If the corresponding frequencies are resonant, that is $\omega_0 \approx \omega_1 + \omega_2$ ($\omega_i = \omega(k = k_i)$), then there can be a strong effect.

1.4: Korteweg-de Vries (KdV) equation

Here we consider the long-wave regime, where $k \to 0$, and assume that we can use the approximate dispersion relation

$$\omega = c_0 k - \beta k^3 , \tag{6}$$

with an error of $O(k^5)$. This translates to an evolution equation

$$u_t + c_0 u_x + \beta u_{xxx} = 0 , \tag{7}$$

where we recall that $-i\omega = \partial/\partial t, ik = \partial/\partial x$ for each Fourier component. The dominant term is $u_t + c_0 u_x \approx 0$, showing that the wave propagates with speed c_0 unchanged, except for the effect of the weak dispersion due to the term u_{xxx}. This small effect needs to be balanced by nonlinearity, and in many physical systems this has the form $\mu u u_x$, for some constant coefficient μ. Thus the model equation takes the form

$$u_t + c_0 u_x + \mu u u_x + \beta u_{xxx} = 0 . \tag{8}$$

This is the famous **Korteweg-de Vries (KdV)** equation, first derived in the water-wave context in 1895, and subsequently found to hold in many physical systems.

1.5: KdV equation, solitons

The KdV equation is, in the reference frame moving with speed c_0 (transform $x \to x - c_0 t$),

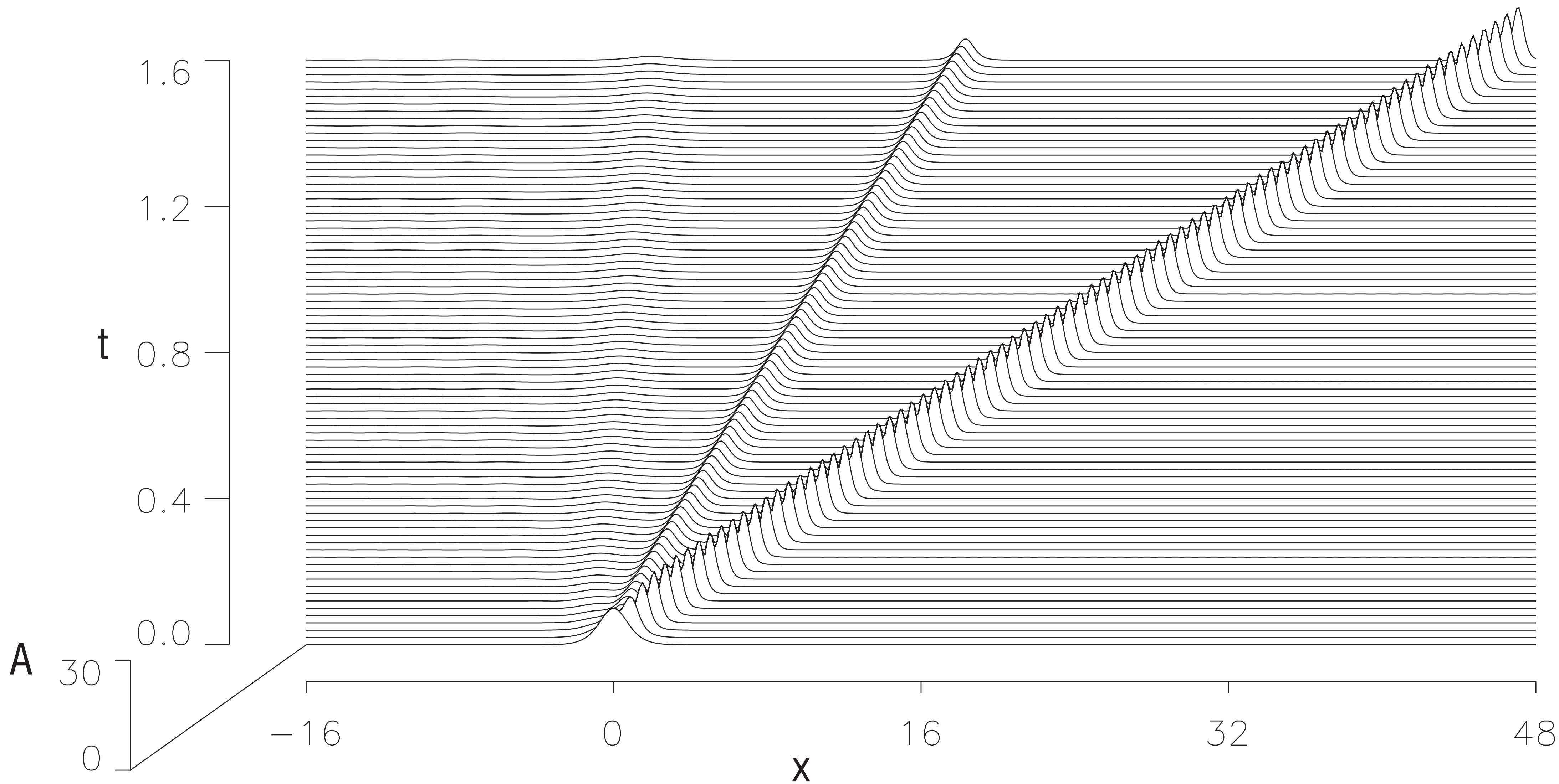
$$u_t + \mu u u_x + \beta u_{xxx} = 0\,. \tag{9}$$

This is an **integrable** equation,a result first established in the 1960's by Kruskal and collaborators. Its principal solutions are **solitons**. A single soliton is the **solitary wave**, an isolated and steadily-propagating pulse, given by

$$u = a\,\mathrm{sech}^2(\gamma(x - Vt))\,, \quad V = \frac{\mu a}{3} = 4\beta\gamma^2\,. \tag{10}$$

This is a one-parameter family of solutions, parametrized by the amplitude a say. The speed V is proportional to the amplitude a and is positive (negative) as $\beta > (<)0$, and is also proportional to the square of the wavenumber γ; thus large waves are thinner and travel faster. They are waves of elevation (depression) when $\mu\beta > (<)0$ Integrability means that the general initial-value problem for a localized initial condition can be solved through the **Inverse Scattering Transform** (IST), with the generic outcome of a finite number of solitons propagating in the positive x-direction, and some dispersing radiation, propagating in the negative x-direction (when $\mu\beta > 0$).

1.6: KdV equation, solitons



The generation of three solitons from a localized initial condition for the KdV equation

$$A_t + 6AA_x + A_{xxx} = 0\,.$$

Harvey's Lectures

1. Introduction to water waves

Derivation of the standard equations for irrotational water waves due to gravity and/or surface tension for an inviscid, incompressible, homogeneous fluid.

3. Hamiltonian formation: water waves

Introduction to the Hamiltonian formulation, and issues such as integrability.

4. Waves in shallow water

Derivation of KdV for water waves by multiple scales; integrability and inverse scattering, comparison with experiments.

7. Oceanographic Applications

The 2004 tsunami, hurricane Katrina (2006), experiments etc.; the periodic solutions of the Kadomtsev-Petviashvili (KP) equation (spatial extension of the KdV equation).

8. The shallow water equations

The nonlinear shallow water equations, method of characteristics, wave breaking, dissipative or dispersive regularization; wave shoaling.

12. Triad (or 3-wave) resonances

Three-wave resonant triad equations, applications to gravity-capillary waves and internal waves.

13. Waves on deep water, I

Modulation (Benjamin-Feir) instability, the NLS model, integrability of NLS in one spatial dimension, envelope solitons.

14. Waves on deep water, II

NLS model in one and two spatial dimensions, recurrence of initial states, downshifting, two-dimensional wave patterns, the role of damping.

19. The Explosive Instability due to 3-wave of 4-wave mixing

Explosive instability due to 3-wave or 4-wave mixing.

20. Potpourri

Topics left over from previous lectures. If time permits: rip currents (in shallow water), rogue waves (in deep water).

Roger’s lectures

2. Introduction

Introduction to the Korteweg-de Vries (KdV) equation as a model for weakly nonlinear long waves, and to the Nonlinear Schrodinger equation (NLS) as a model for weakly nonlinear wave packets.

5. Derivation of the Korteweg-de Vries Equation

Derivation of KdV for weakly nonlinear long internal waves in the ocean.

6. Nonlinear waves in a variable medium

The variable-coefficient KdV equation, and slowly-varying solitary waves.

9. Internal solitary waves in the ocean

Application of the variable-coefficient KdV equation, and related model equations, to the description of large-amplitude internal waves in the coastal ocean.

10. Whitham Modulation Theory

Introduction to the Whitham modulation theory, undular bores, using the context of the KdV equation.

11. Transcritical flow over an obstacle

Description of flow interaction of an obstacle, based on the forced KdV equation.

15. Solitary Waves

Existence of solitary waves, as homoclinic orbits in a spatial dynamical system.

16. General Solitary Waves

Resonance between long and short waves, exponential asymptotics

17. Wave-Mean Flow Interaction, Part I

Application of the Whitham modulation theory to the interaction of nonlinear water waves with currents.

18. Wave-Mean Flow Interaction, Part II

The generalized Lagrangian mean theory, applied to internal waves.

References

- **Lecture slides** (http://www.whoi.edu/page.do?pid=7945)
- **Linear and Nonlinear Waves** (Whitam, 1978)
- **MAGIC lectures on NONLINEAR WAVES:** (http://maths.dept.shef.ac.uk/magic/course.php?id=21)
- **Lecture notes by GFD Fellows** (available on line soon)