1.5: KdV equation, solitons

The KdV equation is, in the reference frame moving with speed c_0 (transform $x \to x - c_0 t$),

$$u_t + \mu u u_x + \beta u_{xxx} = 0 \,. \tag{9}$$

This is an **integrable** equation,a result first established in the 1960's by Kruskal and collaborators. Its principal solutions are **solitons**. A single soliton is the **solitary wave**, an isolated and steadily-propagating pulse, given by

$$u = a\,\mathrm{sech}^2(\gamma(x - Vt))\,, \quad V = \frac{\mu a}{3} = 4\beta\gamma^2 \,. \tag{10}$$

This is a one-parameter family of solutions, parametrized by the amplitude a say. The speed V is proportional to the amplitude a and is positive (negative) as $\beta > (<) 0$, and is also proportional to the square of the wavenumber γ; thus large waves are thinner and travel faster. They are waves of elevation (depression) when $\mu\beta > (<) 0$ Integrability means that the general initial-value problem for a localized initial condition can be solved through the **Inverse Scattering Transform** (IST), with the generic outcome of a finite number of solitons propagating in the positive x-direction, and some dispersing radiation, propagating in the negative x-direction (when $\mu\beta > 0$).