Because the plume is accelerated by buoyancy, the momentum flux changes with z as

$$\begin{aligned}\frac{\partial M}{\partial z} &= \pi b^2 g', && (9)\\ &= \frac{(\pi g' b^2 w)/2 \cdot \pi b^2 w}{(\pi b^2 w^2)/2}, && (10)\\ &= \frac{FQ}{M}. && (11)\end{aligned}$$

We can set $F = F_0$ (constant) because the entrainment not only decrease the buoyancy g' but also increase the flow flux Q. Therefore, the equations of change of Q and M are written as

$$\begin{aligned}\frac{\partial Q}{\partial z} &= 2\sqrt{2}\pi^{\frac{1}{2}}\alpha M^{\frac{1}{2}}, && (12)\\ \frac{\partial M}{\partial z} &= \frac{F_0 Q}{M}. && (13)\end{aligned}$$

Now, we assume Q and M as

$$\begin{aligned}Q &= Q_0(z+z_V)^q, && (14)\\ M &= M_0(z+z_V)^m, && (15)\end{aligned}$$

where z_V is the height of "vertial origin", and Q_0, M_0 are constants. This assumption means that the radius of the plume b increase linearly from 0 at its vertial orign $z = -z_V$. Then (12) and (13) become

$$\begin{aligned}Q_0 q(z+z_V)^{q-1} &= 2\sqrt{2}\pi^{\frac{1}{2}}\alpha M_0^{\frac{1}{2}}(z+z_V)^{\frac{m}{2}}, && (16)\\ M_0 m(z+z_V)^{m-1} &= \frac{F_0 Q_0}{M_0}(z+z_V)^{q-m}. && (17)\end{aligned}$$

From above equations, we obtain

$$\begin{aligned}q &= \frac{5}{3}, && (18)\\ m &= \frac{4}{3}. && (19)\end{aligned}$$

Substituting (18) and 19) into (16) and (17) with some algebra, we obtain

$$\begin{aligned}M_0 &= \left(\frac{9\sqrt{2}\pi^{\frac{1}{2}}\alpha F_0}{10}\right)^{\frac{2}{3}}, && (20)\\ Q_0 &= \frac{6\pi^{\frac{1}{2}}\alpha}{5}\left(\frac{18\pi^{\frac{1}{2}}\alpha F_0}{5}\right)^{\frac{1}{3}}. && (21)\end{aligned}$$